



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE INTERSECTIONS OF A STRAIGHT LINE AND HYPERQUADRIC.

By J. L. COOLIDGE.

In analytic geometry of a fairly elementary sort, we frequently encounter the problem of solving simultaneously a number of linear equations, and a single quadratic equation. In homogeneous form this amounts to solving $n - 1$ linear equations, and one irreducible quadratic in $n + 1$ homogeneous variables, or, to use the jargon of projective geometry, of finding the intersections of a straight line and a hyperquadric in a space of n dimensions.

We all know that the problem can be solved, generally we content ourselves with saying that it is easy, or with writing down the single irrationality that enters into the solution. There do arise cases, however, when we must actually put the work through. We must actually have the coördinates of the intersections of a straight line and a conic, or those of the limiting points of a set of coaxal spheres, or those of the lines that intersect four skew lines in our space. The work of putting through the solution is somewhat repellant, yet when we search in the textbooks we fail to find there any solution in symmetrical form. Such, at least, has been the present writer's experience.* The fact is that every solution must be either unsymmetrical or involve arbitrary quantities dragged in from the outside. Of the two alternatives the latter is certainly preferable, and in the present paper a solution is given which seems to be about as compact and symmetrical as the problem will allow. The final form will be found in formulæ (14) and (15).

Let a point in n space have $n + 1$ homogeneous coördinates

$$x_0 : x_1 : x_2 : \cdots x_n.$$

We suppose that we have $n - 1$ homogeneous linear equations

$$(1) \quad \sum_{i=0}^{i=n} u_i^{(1)} x_i = \sum_{i=0}^{i=n} u_i^{(2)} x_i = \cdots = \sum_{i=0}^{i=n} u_i^{(n-1)} x_i = 0$$

and a single quadratic equation

* A criterion for the reality of the solutions, when the discriminant of the quadric is not 0, is given by Snyder, "Conditions that a line," etc., Bulletin American Math. Soc., vol. 4, 1897, p. 72.

$$(2) \quad \sum_{i,j=0}^{i,j=n} a_{ij} x_i x_j = 0, \quad a_{ij} = a_{ji}.$$

Since, by hypothesis, this quadratic form is unfactorable, the rank of the matrix $|| a_{ij} ||$ is 3 or greater. We seek for the solutions of the equations (1) and (2). We begin with the explanation of certain notations, and the development of useful identities. Let

$$(3) \quad \Delta = \begin{vmatrix} v_0 & v_1 & \cdots & v_n \\ w_0 & w_1 & \cdots & w_n \\ u_0^{(1)} & u_1^{(1)} & \cdots & u_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ u_0^{(n-1)} & u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix},$$

$$(4) \quad p_{ij} = -p_{ji} = \frac{1}{\Delta} \left(\frac{\partial \Delta}{\partial v_i} \frac{\partial \Delta}{\partial w_j} - \frac{\partial \Delta}{\partial v_j} \frac{\partial \Delta}{\partial w_i} \right).$$

The quantities p_{ij} are independent of v and w ; they are the Plücker coördinates of the line in question. They are connected by certain interesting identities

$$(5) \quad 2(p_{ij}p_{kl} + p_{ik}p_{lj} + p_{il}p_{jk}) = \frac{1}{\Delta^2} \begin{vmatrix} \frac{\partial \Delta}{\partial v_i} & \frac{\partial \Delta}{\partial v_j} & \frac{\partial \Delta}{\partial v_k} & \frac{\partial \Delta}{\partial v_l} \\ \frac{\partial \Delta}{\partial w_i} & \frac{\partial \Delta}{\partial w_j} & \frac{\partial \Delta}{\partial w_k} & \frac{\partial \Delta}{\partial w_l} \\ \frac{\partial \Delta}{\partial v_i} & \frac{\partial \Delta}{\partial v_j} & \frac{\partial \Delta}{\partial v_k} & \frac{\partial \Delta}{\partial v_l} \\ \frac{\partial \Delta}{\partial w_i} & \frac{\partial \Delta}{\partial w_j} & \frac{\partial \Delta}{\partial w_k} & \frac{\partial \Delta}{\partial w_l} \end{vmatrix} = 0,$$

$$(6) \quad \begin{vmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ p_{ji} & 0 & p_{jk} & p_{jl} \\ p_{ki} & p_{kj} & 0 & p_{kl} \\ p_{li} & p_{lj} & p_{lk} & 0 \end{vmatrix} = 0,$$

$$(7) \quad \begin{vmatrix} p_{ij} & p_{ik} & p_{il} \\ 0 & p_{jk} & p_{jl} \\ p_{kj} & 0 & p_{kl} \end{vmatrix} = 0.$$

Consider next the determinant

$$(8) \quad \begin{vmatrix} 0 & p_{ij} & p_{ik} & \cdots & p_{iy} \\ p_{ji} & 0 & p_{jk} & \cdots & p_{jy} \\ p_{ki} & p_{kj} & 0 & \cdots & p_{ky} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{yi} & p_{yj} & p_{yk} & \cdots & 0 \end{vmatrix}.$$

This is skew symmetric, and hence vanishes when its order is odd. It is also equal to zero when the order is four, as we see by (6). In general, when the order is even it is the perfect square of a function that is linear in a system of like determinants of an order two less,* and so vanishes if its order is even and greater than four. In other words, this determinant vanishes in every case when the order is other than 2. Likewise, consider

$$(9) \quad \begin{vmatrix} p_{ij} & p_{ik} & \cdots & p_{iy} & p_{iz} \\ 0 & p_{jk} & \cdots & p_{jy} & p_{jz} \\ p_{kj} & 0 & \cdots & p_{ky} & p_{kz} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{yj} & p_{yk} & \cdots & 0 & p_{yz} \end{vmatrix}.$$

If we expand in terms of the elements of the last column, each cofactor is either of the present form and lower order, or of the form (8). The latter can be at once rejected, and the former gradually whittled down to (7). Hence (9) vanishes when its degree is above 2. The last two results can be combined in a single identity

$$(10) \quad \begin{vmatrix} z_0 & p_{ij} & p_{ik} & \cdots & p_{iy} \\ z_1 & 0 & p_{jk} & \cdots & p_{jy} \\ z_2 & p_{kj} & 0 & \cdots & p_{ky} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_r & p_{yj} & p_{yk} & \cdots & 0 \end{vmatrix} \equiv 0, \quad r > 2.$$

We have now, fortunately, finished our preliminary work, and may proceed at once to the problem in hand. Let us suppose, as a first case, that equations (1) and (2) have been so far modified by a collineation of space, that the latter reduces to the form

$$\sum_{i=0}^{i=n} a_i x_i^2 \equiv \sum_{i=0}^{i=n} (a_i x_i) x_i = 0.$$

This latter form appears as quasi-linear in (x) . Adding this to the linear

* Conf. Kowalewski, "Determinantentheorie," Leipzig, 1909, pp. 134 ff.

equations (1) we have n linear equations in $n + 1$ homogeneous variables. Solving these in the usual way we find

$$(11) \quad -px_k = \frac{\partial}{\partial t_k} \begin{vmatrix} t_0 & t_1 & \cdots & t_n \\ u_0^{(1)} & u_1^{(1)} & \cdots & u_n^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ u_0^{(n-1)} & u_1^{(n-1)} & \cdots & u_n^{(n-1)} \\ a_0x_0 & a_1x_1 & \cdots & a_nx_n \end{vmatrix}.$$

This time we have $n + 1$ really linear equations in as many variables, the condition of compatibility gives the following equation in p

$$\begin{vmatrix} p & a_1p_{01} & a_2p_{02} & \cdots & a_np_{0n} \\ a_0p_{10} & p & a_2p_{12} & \cdots & a_np_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_0p_{n0} & a_1p_{n1} & a_2p_{n2} & \cdots & p \end{vmatrix} = 0.$$

The coefficient of each power of p below the three highest is of the form (8) and so is equal to zero. We have, thus,

$$(12) \quad p^2 + \frac{1}{2} \sum_{i,j=0}^{i,j=n} a_i a_j p_{ij}^2 = 0.$$

If we take p from this equation, our equations (11) are consistent. We might omit any one of them, and find the ratios of our variables from the remainder. Such a set of solutions would, however, be unsymmetrical, owing precisely to the rejection of one equation. It is preferable to multiply through the solutions obtained by rejecting the i th equation by an arbitrary multiplier z_i and take the sum. We thus get x_0 , for instance

$$\rho x_0 = \begin{vmatrix} z_0 & a_1p_{01} & a_2p_{02} & \cdots & a_np_{0n} \\ z_1 & p & a_2p_{12} & \cdots & a_np_{1n} \\ z_2 & a_1p_{21} & p & \cdots & a_np_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ z_n & a_1p_{n1} & a_2p_{n2} & \cdots & p \end{vmatrix}.$$

Expanding in terms of p , we see, by (10) that the coefficients of all but the three highest powers of p are zero. Dividing through by p^{n-2}

$$\rho x_0 = z_0 p^2 - p \sum_{i=0}^{i=n} a_i z_i p_{0i} + \frac{1}{2} \sum_{i,j=0}^{i,j=n} a_i a_j p_{ij} (z_0 p_{ij} + z_i p_{j0} + z_j p_{0i}).$$

Remembering (12) and putting

$$(13) \quad \begin{aligned} P_{ki} &= \sum_{j=0}^{j=n} a_j p_{kj} p_{ji}, \\ \rho x_k &= \sum_{i=0}^{i=n} a_i z_i (P_{ki} - p p_{ki}). \end{aligned}$$

It only remains to rewrite this in invariant form when the hyperquadric has the general equation (2). We write first

$$(14) \quad p^2 + \frac{1}{2} \sum_{l, i=0}^{l, i=n} \sum_{m, j=0}^{m, j=n} a_{li} a_{mj} p_{lm} p_{ij} = 0.$$

Then, by frequent use of (5) we find

$$(15) \quad \rho x_k = \sum_{l, i=0}^{l, i=n} \left[a_{li} z_i \left(\sum_{m, j=0}^{m, j=n} a_{mj} p_{lm} p_{kj} + p p_{lk} \right) \right].$$

CAMBRIDGE, MASS.,
December, 1916.